

Quantum aging in mean-field models

Leticia F. Cugliandolo^{*†} and Gustavo Lozano^{**††}

^{*} *Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 rue Lhomond, F-75231 Paris Cedex 05, France*
^{**} *Laboratoire de Physique Théorique et Hautes Energies*

Univ. Paris VI and VII, 5 ème étage, Tour 24, 4 Place Jussieu, 75252 Paris Cedex 05, France

^{**} *Division de Physique Théorique, IPN, Université de Paris XI, F-91400, Orsay, France*

We study the real-time dynamics of quantum models with long-range interactions coupled to a heat-bath within the closed-time path-integral formalism. We show that quantum fluctuations depress the transition temperature. In the subcritical region there are two asymptotic time-regimes with (i) stationary, and (ii) slow aging dynamics. We extend the quantum fluctuation-dissipation theorem to the nonequilibrium case in a consistent way with the notion of an effective temperature that drives the system in the aging regime. The classical results are recovered for $\hbar \rightarrow 0$.

PACS numbers: 05.20.-y, 02.50.Ey, 05.40.+j

The dynamics of nonequilibrium systems is being intensively studied now. Notably, glassy systems below their critical temperature have a very slow evolution with nonstationary dynamics [1]. Several theoretical ideas [2] are used to describe it, namely, scaling arguments, phase space models, analytical solutions to mean-field models and numerical simulations. The analysis of simple mean-field models (with long-range interactions) has provided a general scenario [3,4] that is now being verified numerically for more realistic models [5,6]. All these studies concern classical systems.

Recent experiments [7] have motivated a renewed interest on the effect of quantum fluctuations (QF) on glassy systems. Up to present, theoretical studies have focused on how QF affect their *equilibrium* properties [8–11].

Since glasses below T_g are not expected to reach equilibrium in experimentally accessible times, it is important to devise a method to understand the influence of QF on the trully nonequilibrium *real-time* dynamics of this type of systems. Intuitively, one expects QF to only affect the short-time dynamics; however, they are also expected to act as thermal fluctuations. It is then not clear whether QF would destroy glassiness or modify it drastically.

Our aims are (i) to present a formalism suited to study the real-time dynamics of a nonlinear, possibly disordered, model in contact with a bath; (ii) to propose a framework to study its dynamics that could be also applicable to more realistic, finite dimensional, models; (iii) to show that below a critical line QF do not destroy the nonequilibrium effects of the glassy phase and that QF add up to an effective temperature T_{EFF} ; (iv) to prove that T_{EFF} is non-zero even at zero bath temperature, that it drives the dynamics at late epochs and that it makes the dynamics appear classical in that time-regime. A longer account of our results will appear elsewhere.

The real-time dynamics of a quantum system is described with a closed-time path-integral generating functional (CTP-GF) [12]. We choose a set of noninteracting harmonic oscillators with an adequate distribution of frequencies as a bath, and a linear interaction between bath

and system [13]. The bath variables are next integrated out and the effect of the bath manifests in the effective action through two non-local kernels associated to dissipation (η) and noise (ν). If the model is disordered, one needs to compute averaged expectation values. However, the CTP-GF without sources is independent of the realization of disorder and one can hence avoid, in a quantum dynamical calculation, the introduction of replicas. In addition, when $\hbar \rightarrow 0$, the CTP-GF yields the classical Martin-Siggia-Rose one.

For the sake of concreteness, we study the p spin-glass

$$H_J[\phi] = \frac{1}{2mN} \sum_{i=1}^N \Pi_i^2 + \sum_{i_1, \dots, i_p} J_{i_1 \dots i_p} \phi_{i_1} \dots \phi_{i_p} \quad (1)$$

with Π_i the canonical momenta, $[\Pi_i, \phi_j] = -i\hbar \delta_{ij}$. The multispin interactions $J_{i_1 \dots i_p}$ are taken from a Gaussian distribution with zero mean and variance $\tilde{J}^2 p! / (2N^{p-1})$ and $\sum_{i=1}^N [\langle \phi_i^2(t) \rangle]_J = N$, $\forall t$. Square brackets denote an average over disorder and $\langle \bullet \rangle$ the average over temporal histories. The quantum mean-field equations follow from a saddle-point approximation and involve the symmetrized auto-correlation $NC(t, t_w) \equiv [\langle \phi(t) \phi(t_w) + \phi(t_w) \phi(t) \rangle]_J$ and the response to an infinitesimal perturbation h applied at time t_w , $NR(t, t_w) \equiv \delta[\langle \phi(t) \rangle / \delta h(t_w)]_J|_{h=0}$. We define the vertex and self-energy:

$$D(t, t_w) = \tilde{D}(t, t_w) + 2\hbar\nu(t - t_w) \\ \Sigma(t, t_w) = \tilde{\Sigma}(t, t_w) - 4\eta(t - t_w)$$

respectively. $\tilde{\Sigma}$ and \tilde{D} are functions of C and R that follow either from the average over disorder or from approximations of the nonlinear interactions in the model (e.g. mode-coupling approximations). For the p -spin model and $t \geq t_w$ they read

$$\tilde{D}(t, t_w) + \frac{i\hbar}{2} \Sigma(t, t_w) = \frac{p\tilde{J}^2}{2} \left(C(t, t_w) + \frac{i\hbar}{2} R(t, t_w) \right)^{p-1}$$

The terms associated to the bath are

$$\nu(t - t_w) = \int_0^\infty d\omega I(\omega) \coth(\beta\hbar\omega/2) \cos(\omega(t - t_w)) ,$$

$$\eta(t - t_w) = \theta(t - t_w) \int_0^\infty d\omega I(\omega) \sin(\omega(t - t_w)) .$$

We chose an Ohmic distribution of oscillator frequencies $I(\omega) = (M\gamma_o/\pi) \omega \exp(-|\omega|/\Lambda)$ with Λ a cut-off and $M\gamma_o$ a constant that plays the rôle of a friction coefficient. Their particular form depends on the choice of bath. The equations, for a random initial condition, are

$$R(t, t_w) = G_o(t, t_w) + \int_0^t \int_0^t du dv G_o(t, u) \Sigma(u, v) R(v, t_w)$$

$$C(t, t_w) = \int_0^t du \int_0^{t_w} dv R(t, u) D(u, v) R(t_w, v) \quad (2)$$

with $G_o^{-1}(t, u) = \delta(t - u) (m\partial_u^2 + \mu(t))$ the propagator and $\mu(t)$ either a Lagrange multiplier enforcing a spherical constraint or the strength of a quadratic term. In the first case it is determined by the ‘gap’ equation: $\mu(t) = \int_0^t du \Sigma(t, u) C(t, u) + D(t, u) R(t, u) - m \partial_t C(t, t_w)|_{t_w \rightarrow t^-}$. Causality implies $R(t, t_w) = \Sigma(t, t_w) = 0$ if $t_w > t$. The inertial term imposes continuity on the equal-times correlator $\lim_{t_w \rightarrow t^-} \partial_t C(t, t_w) = \lim_{t_w \rightarrow t^+} \partial_t C(t, t_w) = 0$, and $R(t, t) = 0$, $\lim_{t_w \rightarrow t} \partial_t R(t, t_w) = 1/m$. In what follows we set $t \geq t_w$. The coupling to the bath implies dissipation; if $M\gamma_o \neq 0$ the energy density \mathcal{E} of the system decreases. One can then envisage to switch-off the coupling (set $M\gamma_o = 0$) when \mathcal{E} reaches a desired value and follow the subsequent evolution at constant \mathcal{E} . This would be useful to further understand the energy landscape. Here, we keep $M\gamma_o \neq 0$ for all times, reparametrize time according to $t \rightarrow M\gamma_o t$ and transform \hbar in a free parameter $\hbar \rightarrow M\gamma_o \hbar$. Consistently, $C \rightarrow C$, $R \rightarrow R/(M\gamma_o)$, $m \rightarrow (M\gamma_o)^2 m$, $\tilde{J} \rightarrow \tilde{J}$, $\beta \rightarrow \beta$, $\Lambda \rightarrow \Lambda/(M\gamma_o)$ and the units are $[C] = [R] = [\hbar] = 1$, $[m] = [\beta] = [1/\tilde{J}] = [1/\Lambda] = [t]$.

We focus on $p > 2$, since $p = 2$ needs a special treatment [16]. Since Eqs. (2) are causal, we construct its solution numerically, step by step in t , spanning $0 \leq t_w \leq t$. The numerical and analytical studies yield:

(i) *Quantum mode-coupling equations.* For \hbar and T above a critical line, there is a finite equilibration time τ_{EQ} after which equilibrium dynamics sets in. The solution satisfies invariance under time-translations (TTI) and the quantum fluctuation-dissipation theorem (QFDT). It is a ‘paramagnetic/liquid’ phase. A TTI-QFDT ansatz yields

$$R(\omega) = \frac{1}{-m\omega^2 + \mu_\infty - \Sigma(\omega)} , \quad (3)$$

$$C(\omega) = D(\omega) |R(\omega)|^2 . \quad (4)$$

with the couples R and C , and Σ and D , verifying QFDT:

$$R(\tau) = \frac{2i}{\hbar} \theta(\tau) \int \frac{d\omega}{2\pi} e^{-i\omega\tau} \tanh\left(\frac{\beta\hbar\omega}{2}\right) C(\omega) \quad (5)$$

($\tau = t - t_w$). Equations (3)-(4) are the quantum version of the mode-coupling equations used to describe supercooled-liquids [14]. Away from the critical line, C and R decay to zero very fast, with oscillations. Approaching the critical line $T_d(\hbar_d)$, the decay slows down and, if $T_d(\hbar_d) \neq 0$, a plateau develops in C . At the critical line, the length of the plateau tends to infinity. The quantum critical point ($T_d = 0$, $\hbar_d \neq 0$) is discussed below.

(ii) *Dynamics in the glassy phase.* For \hbar and T below the critical line, $\tau_{\text{EQ}} \rightarrow \infty$ (as a function of N): times are always finite with respect to τ_{EQ} . The system does not reach equilibrium. This is a ‘spin-glass/glass’ phase. There are two time regimes with different behaviours according to the relative value of $t - t_w$ and a characteristic (model-dependent) time $\mathcal{T}(t_w)$:

If $t - t_w \leq \mathcal{T}(t_w)$ ($C(t, t_w) \geq q$) the dynamics is *stationary*; TTI and QFDT hold. In other words,

$$q + C_{\text{ST}}(t - t_w) = \lim_{t_w \rightarrow \infty} C(t, t_w) \quad (6)$$

and $R_{\text{ST}}(t - t_w) = \lim_{t_w \rightarrow \infty} R(t, t_w)$. The correlation approaches a plateau q since $\lim_{t-t_w \rightarrow \infty} C_{\text{ST}}(t - t_w) = 0$. The response approaches zero, $\lim_{t-t_w \rightarrow \infty} R_{\text{ST}}(t - t_w) = 0$. The equations for C_{ST} and R_{ST} are identical to Eqs.(3)-(4) apart from contributions to μ_∞ coming from the aging regime. C_{ST} and R_{ST} are linked through Eq.(5).

If $t - t_w > \mathcal{T}(t_w)$ ($C(t, t_w) < q$) the dynamics is *nonstationary*; TTI and FDT do not hold and there is quantum aging. The correlation decays from q to 0 and we call it $C_{\text{AG}}(t, t_w)$. The decay of C becomes *monotonous* in the aging regime; the properties of monotonous two-time correlation functions derived in Ref. [4] imply:

$$C_{\text{AG}}(t, t_w) = j^{-1} \left(\frac{h(t_w)}{h(t)} \right) , \quad (7)$$

with $h(t)$ a growing function of time. For the p -spin model, j^{-1} is the identity.

The system has a weak long-term memory, meaning that the response tends to zero *but* its integral over a growing time-interval is *finite*. The QFDT is more involved than the classical one since it is non-local (Eq.(5)). In the nonequilibrium case, we generalize it to

$$R(t, t_w) = \frac{2i}{\hbar} \theta(t - t_w) \int_{-\infty}^\infty \frac{d\omega}{2\pi} \exp(-i\omega(t - t_w))$$

$$\times \tanh\left(\frac{X(t, t_w)\beta\hbar\omega}{2}\right) C(t, \omega) \quad (8)$$

$$C(t, \omega) \equiv 2\text{Re} \int_0^t ds \exp(i\omega(t - s)) C(t, s) \quad (9)$$

with $X(t, t_w)$ a function of t and t_w . If the evolution is TTI and $X(t, t_w) = 1$ for all times, Eq.(8) reduces to Eq.(5). Interestingly enough, $T_{\text{EFF}} \equiv T/X(t, t_w)$ acts as an effective temperature in the system [15]. For a model with two two-time sectors we propose

$$X(t, t_w) = \begin{cases} X_{\text{ST}} = 1 & \text{if } t - t_w \leq \mathcal{T}(t_w) \\ X_{\text{AG}}(\hbar, T) & \text{if } t - t_w > \mathcal{T}(t_w) \end{cases}.$$

with X_{AG} a non-trivial function of \hbar and T . When t and t_w are widely separated, the integration over ω in Eq.(8) is dominated by $\omega \sim 0$. Therefore, the factor $\tanh(X_{\text{AG}}(t, t_w)\beta\hbar\omega/2)$ can be substituted by $X_{\text{AG}}\beta\hbar\omega/2$ (even at $T = 0$ if $X_{\text{AG}}(\hbar, T) = x(\hbar)T$ when $T \sim 0$). Hence,

$$R_{\text{AG}}(t, t_w) \sim \theta(t - t_w) X_{\text{AG}} \beta \partial_{t_w} C_{\text{AG}}(t, t_w) \quad (10)$$

and one recovers, in the aging regime, the *classical* modified FDT [3,11]. Numerically (and experimentally), it is simpler to check it using the integrated response $\chi(t, t_w) = \int_{t_w}^t dt'' R(t, t'')$ over a large time window [4,6], that reads

$$\chi(t, t_w) = \begin{cases} \int_0^{t-t_w} d\tau' R_{\text{ST}}(\tau') \\ \int_0^\infty d\tau' R_{\text{ST}}(\tau') + \frac{X_{\text{AG}}}{T} (q - C_{\text{AG}}(t, t_w)) \end{cases}$$

the first line holds for $C(t, t_w) > q$ while the second holds for $C(t, t_w) < q$. Remarkably, the correlation and the violation of QFDT are given by similar expressions to the classical – though the values of q and X_{AG} depend on \hbar . In this sense, we say that $\mathcal{T}(t_w)$ acts a time-dependent ‘decoherence time’, beyond which the nonequilibrium regime is ‘classical’.

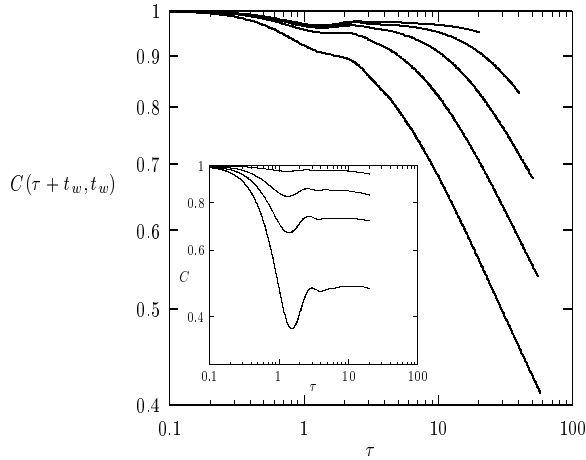


FIG. 1. The correlation function $C(\tau + t_w, t_w)$ vs τ for the $p = 3$ SG model, $\Lambda = 5$, $\tilde{J} = 1$, $m = 1$, $T = 0$ and $\hbar = 0.1$. The waiting times are, from bottom to top, $t_w = 2.5, 5, 10, 20, 40$. $q \sim 0.97$. In the inset, the same curves for $t_w = 40$ and, from top to bottom, $\hbar = 0.1, 0.5, 1, 2$.

An ansatz like Eq.(7) for C_{AG} combined with Eq.(10) for the FDT violation solves the equations in the aging regime. It yields

$$1 = (R_{\text{ST}}(\omega = 0))^2 \tilde{J}^2 p(p-1) q^{p-2} / 2 \quad (11)$$

$$X_{\text{AG}}/T = R_{\text{ST}}(\omega = 0)(p-2)/q \quad (12)$$

that become $\tilde{J}^2 p(p-1)/2q^{p-2}(1-q)^2 = 2T^2$ and $X_{\text{AG}} = (p-2)(1-q)/q$ when $\hbar \rightarrow 0$ [3].

QF depress the critical temperature. The transition line $T_d(\hbar_d)$ ends at a quantum critical point ($T_d = 0, \hbar_d \neq 0$). Equations (11)-(12) indicate that when the transition occurs at $T_d(\hbar_d) \neq 0$ it is first-order with $X_{\text{AG}} \rightarrow 1$, $q \rightarrow q_d \neq 0$ and a finite linear stationary susceptibility $R_{\text{ST}}(\omega = 0) < +\infty$ (as in the classical limit). On this line, $q_d \propto T_d^{2/p}$ and $q_d \rightarrow 0$ for $T_d \rightarrow 0$. At the quantum critical point, Eqs.(11)-(12) suggest a second-order transition; if q_d tends to zero as $q_d \sim (\hbar_d - \hbar)^{\alpha p/2}$, then $X_{\text{AG}}/T \sim (\hbar_d - \hbar)^{-\alpha}$, and $R_{\text{ST}}(\omega = 0) \sim (\hbar_d - \hbar)^{\alpha(1-p/2)}$ diverges when $p > 2$ (α is positive).

Let us now discuss the numerical results. In all figures $p = 3$, $\Lambda = 5$, $\tilde{J} = 1$ and $m = 1$. We use $T = 0$ and $\hbar = 0.1$ to illustrate the dynamics in the glassy phase. We also discuss the dependence upon T and \hbar .

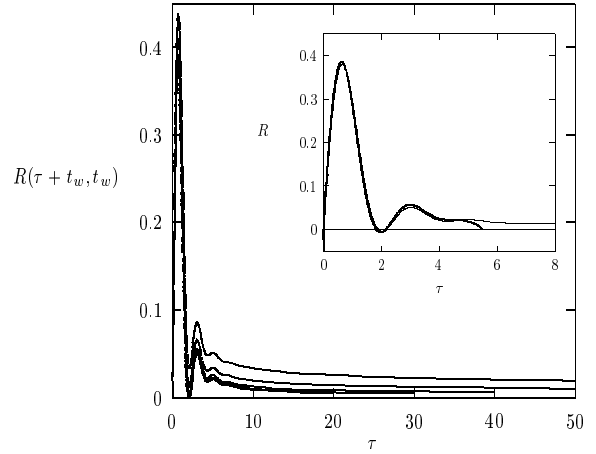


FIG. 2. The response function for the same model as above. The waiting-times increase from top to bottom. In the inset, check of FDT in the stationary regime. The full line is $R(t, t_w)$ for $t = 40$ fixed and $t_w \in [0, 40]$. The dots are obtained from Eq.(8) with $X_{\text{ST}} = 1$, using the numerical data for $C_{\text{ST}}(t - t_w) = C(t, t_w) - q$ ($q \sim 0.97$, see Fig.1). In both cases the response is plotted against $t - t_w$.

In Figs. 1 and 2 we show the correlation $C(\tau + t_w, t_w)$ (log-log plot) and the response $R(\tau + t_w, t_w)$ (linear plot) vs the subsequent time τ . These plots demonstrate the existence of the stationary and aging regimes. For $t - t_w < \mathcal{T}(t_w)$ (e.g. $\mathcal{T}(40) \sim 5$) TTI and FDT are established while beyond $\mathcal{T}(t_w)$ they break down. For $\hbar = 0.1$ the plateau in C is at $q \sim 0.97$. C oscillates around q but is monotonous when it goes below it. In the inset we present the dependence of q on \hbar for $T = 0$. QF generate a $q < 1$ such that the larger \hbar the smaller q . The addition of thermal fluctuations has a similar effect, the larger T , the smaller q . In order to check FDT in the stationary regime, in the inset of Fig. 2 we compare $R(t, t_w)$ from the numerical algorithm for $t = 40$

fixed and $t_w \in [0, 40]$ (full line) with $R(t, t_w)$ from Eq.(8) with $X = 1$ using $C_{ST}(t - t_w) = C(t, t_w) - q$, $q \sim 0.97$ obtained from the algorithm (dots). The accord is very good if $t - t_w \leq T(t_w) \sim 5$.

In Fig.3 we plot the integrated response χ vs. C in a parametric plot. When $C < q \sim 0.97$ the χ vs C curve approaches a straight line of *finite* slope $1/T_{\text{EFF}} = X_{\text{AG}}/T$ that we estimate from Eqs.(11) and (12) as $X_{\text{AG}}/T \sim 0.60$. The dots are a guide to the eye representing it.

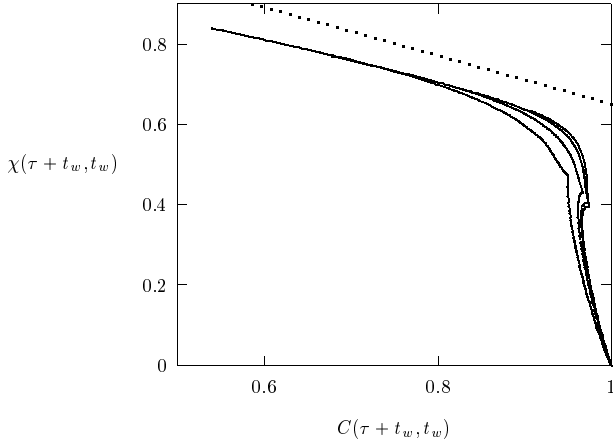


FIG. 3. The integrated response $\chi(\tau + t_w, t_w)$ vs. the symmetrized auto-correlation function $C(\tau + t_w, t_w)$ in a parametric plot. The curves correspond to $t_w = 10, 20, 30, 40$. $T = 0$ and $\hbar = 0.1$. The dots are a guide to the eye and represent the analytic result $X_{\text{AG}}/T \sim 0.60$ for the slope of the asymptotic curve in the aging regime $C < q \sim 0.97$.

To summarize, this formalism provides a clear framework to study the nonequilibrium regime and/or the eventual approach to equilibrium of quantum systems. It circumvents the difficulties of performing numerical simulations in real-time and predicts the existence and the structure of a rich nonequilibrium regime for glassy systems even in the presence of QF. Based on the success of mean-field like classical glassy models to describe some aspects of the dynamics of realistic glassy systems (see [2,14] and references therein) one can expect that quantum mean-field models of the type considered here do also capture important features of the real-time dynamics of real systems.

ACKNOWLEDGEMENTS. We wish to thank T. Evans, A. Georges, D. Grempel, T. Giamarchi, J. Kurchan, P. Le Doussal, R. Revers and M. Rozenberg for useful discussions. G.L is supported by the EC grant N ERBFMBICT 961226.

[†] E-mail address: leticia@physique.ens.fr

^{††} E-mail addresses: lozano@ipno.in2p3.fr

- [1] L. C. E. Struik; *Physical Aging in Amorphous Polymers and Other Materials* (Elsevier, 1978). L. Lundgren, *et al* *Phys. Rev. Lett* **51**, 911 (1983); E. Vincent *et al.* in *Sitges Conference on Glassy Systems*, ed. M. Rubí, (Springer-Verlag 1996).
- [2] Recent progress in theoretical classical glassy models is reviewed in J-P Bouchaud *et al* in *Spin-glasses and random fields*, A. P. Young ed. (World Scientific, 1997), cond-mat/9702070.
- [3] L. F. Cugliandolo and J. Kurchan, *Phys. Rev. Lett.* **71**, 173 (1993); *Phil. Magaz.* **B71**, 50 (1995)
- [4] L. F. Cugliandolo and J. Kurchan, *J. Phys.* **A27**, 5749 (1994).
- [5] H. Rieger; *Ann Rev. of Comp. Phys.* II, D. Stauffer ed. (World Scientific, 1995). E. Andrejew and J. Baschnagel; *Physica* **A233**, 117 (1996). W. Kob and J-L Barrat; *Phys. Rev. Lett.* **78**, 4581 (1997).
- [6] S. Franz and H. Rieger, *J. Stat. Phys.* **79**, 749 (1995); G. Parisi, *Phys. Rev. Lett.* **79**, 3660 (1997); E. Marinari *et al*, cond-mat/9710120; A. Barrat; cond-mat/9710069.
- [7] W. Wu *et al.*, *Phys. Rev. Lett.* **67**, 2076 (1991); *ibid* **71**, 1919 (1993); J. Mattsson, *Phys. Rev. Lett.* **75**, 1678 (1995); D. Bitko *et al*, *ibid* 1679 (1995); R. Pirc, B. Tadic and R. Blinc; *Z. Phys.* **B61**, 69 (1985).
- [8] S. Sachdev, *Physics World* **7**, 25 (1995), cond-mat/9705266; H. Rieger and A. P. Young, *Lecture Notes in Physics* (Springer-Verlag, 1996), cond-mat/9607005; R. N. Bhatt; in ‘Spin-glasses and random fields’, A. P. Young ed., (World Scientific, 1997).
- [9] D. S. Fisher, *Phys. Rev. Lett.* **69**, 534 (1992); *Phys. Rev.* **B51**, 6411 (1995); F. Igloi and H. Rieger, *Phys. Rev. Lett.* **78** (1997) 2473 and cond-mat/9709260.
- [10] A. J. Bray and M. A. Moore; *J. Phys.* **C13**, L655 (1980); V Dobrosavljevic and D Thirumalai, *J. Phys.* **A23**, L767 (1990); Y. Y. Godschmidt, *Phys. Rev.* **E53**, 343 (1996); T. K. Kopeć, *Phys. Rev.* **B54**, 3367 (1996); Th. Nieuwenhuizen and F. Ritort, cond-mat/9706244.
- [11] See T. Giamarchi and P. Le Doussal, *Phys. Rev.* **B53** 15206 (1996) for the intriguing and yet not fully understood question of the determination of X within quantum replica theory.
- [12] J. Schwinger, *J. Math. Phys.* **2**, 407 (1961); L. V. Keldysh, *Zh. Eksp. Teor. Fiz.* **47**, 1515 (1964); *Sov. Phys JETP* **20**, 235 (1965); G. Zhou *et al*, *Phys. Rep.* **118**, 1 (1985).
- [13] R. P. Feynmann and F. L. Vernon, *Ann Phys.* **24**, 118 (1963); A. Caldeira and A. Legget, *Phys Rev.* **A31**, 1059 (1985)
- [14] W. Götze and L. Sjögren, *Rep. Prog. Phys.* **55**, 241 (1992).
- [15] L. F. Cugliandolo, J. Kurchan and L. Peliti, *Phys. Rev.* **E55**, 3898 (1997).
- [16] The $p = 2$ model is related to $O(N)$ quantum field theories in the large N limit (see Ref. [2]). The dynamics of the latter receives now much attention in connection with cosmology; see. e.g., A. Linde; hep-th/9410082. D. Boyanovsky, H. J. de Vega and R. Holman; hep-ph/9701304.